

# Existence and Uniqueness of Normalized Solutions for the Kirchhoff equation

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## Abstract

For a class of Kirchhoff functional, we first give a complete classification with respect to the exponent  $p$  for its  $L^2$ -normalized critical points, and show that the minimizer of the functional, if exists, is unique up to translations. Secondly, we search for the mountain pass type critical point for the functional on the  $L^2$ -normalized manifold, and also prove that this type critical point is unique up to translations. Our proof relies only on some simple energy estimates and avoids using the concentration-compactness principles. These conclusions extend some known results in previous papers.

**MSC:**35J20; 35J60

**Keywords:**  $L^2$ -normalized critical point; Kirchhoff equation; Uniqueness

## 1 Introduction

In this paper, we study the following Kirchhoff equation

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u - |u|^p u = \mu u \text{ in } \mathbb{R}^N, \quad (1.1)$$

where  $a, b > 0$ ,  $1 \leq N \leq 3$ , and  $0 < p < 2^* - 2$  with  $2^* = +\infty$  if  $N = 1, 2$ , or  $2^* = 6$  if  $N = 3$ . Equation (1.1) is related to the stationary solutions of

$$u_{tt} - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u = f(x, u), \quad (1.2)$$

where  $f(x, u)$  is a general nonlinearity. The problem (1.2) was proposed by Kirchhoff [8] and models free vibrations of elastic strings by taking into account the changes in length of the string produced by transverse vibrations. Comparing with the semilinear equations (i.e., setting  $b = 0$  in above two equations), it is much more challenge and interesting to investigate equations (1.1) and (1.2) in view of the existence of the nonlocal term  $\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u$ .

After the pioneering work of [10] and [9], much attention was paid to these above two equations. For instance, replacing the term  $|u|^p u$  with a general nonlinearity  $f(x, u)$ , there are many results on the existence of solutions for equation (1.1), one can refer [2, 4, 5] and the references therein. Equation (1.1) can be viewed as an eigenvalue problem by taking  $\mu$  as an unknown Lagrange multiplier. From this point of view, one can solve (1.1) by studying some constrained variational problems and obtain normalized solutions. Motivated by the works of [1, 12], we consider the following minimization problem:

$$I(c) := \inf_{u \in S_c} E(u) \quad (1.3)$$

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where

$$E(u) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{1}{p+2} \int_{\mathbb{R}^N} |u|^{p+2} dx, \quad (1.4)$$

and

$$S_c := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c^2\}.$$

**Remark 1.1.** If  $u \in S_c$  is a minimizer of problem (1.3), then there exists  $\mu \in \mathbb{R}$  such that  $E'(u) = \mu u$ , namely,  $u \in S_c$  is a solution of (1.1) for some  $\mu \in \mathbb{R}$ . Therefore, it is natural to obtain normalized solutions by investigating problem (1.3).

Ye in [12] proved that when  $0 < p < \frac{8}{N}$ , there exists  $c_p^* > 0$  such that (1.3) has minimizers if and only if  $c > c_p^*$  and  $0 < p \leq \frac{4}{N}$ , or  $c \geq c_p^*$  and  $\frac{4}{N} < p < \frac{8}{N}$ , where  $c_p^*$  is given by

$$c_p^* = \begin{cases} 0, & 0 < p < \frac{4}{N}; \\ a^{\frac{4}{N}} |Q|_{L^2}, & p = \frac{4}{N}; \\ \inf\{c \in (0, +\infty) : I(c) < 0\}, & \frac{4}{N} < p < \frac{8}{N}; \end{cases} \quad (1.5)$$

and  $Q(|x|)$  is the unique (up to translations) radially symmetric positive solution of the following equation in  $H^1(\mathbb{R}^N)$  :

$$-\Delta u + \frac{4+2p-Np}{Np} u - \frac{4}{Np} |u|^p u = 0, \quad 0 < p < 2^* - 2. \quad (1.6)$$

While for the case of  $\frac{8}{N} \leq p < 2^* - 2$ , problem (1.3) cannot be attained. Recently, Guo and Wang in [3] gave the explicit form of  $c_p^*$  for  $p \in (\frac{4}{N}, \frac{8}{N})$ . We note that the arguments of [12] mainly depend on the application of the concentration-compactness principle. By ruling out the cases of *vanishing* and *dichotomy*, the author obtained the compactness of a minimizing sequence.

In what follows, by observing the special form of the functional (1.4), we intend to give a new proof for the above results for problem (1.3) in a simple way, where only technical energy estimates are involved and the concentration-compactness principle is avoided. Especially, our arguments also show that the minimizer of (1.4), if exists, is unique and must be a scaling of  $Q(x)$ . Before stating our main results, we first recall that, combining with the Pohozaev and Nehari identity,  $Q(x)$  satisfies

$$\int_{\mathbb{R}^N} |\nabla Q(x)|^2 dx = \int_{\mathbb{R}^N} |Q(x)|^2 dx = \frac{2}{p+2} \int_{\mathbb{R}^N} |Q(x)|^{p+2} dx. \quad (1.7)$$

Moreover,  $Q$  is an optimizer of the following sharp Gagliardo-Nirenberg inequality [11]

$$\int_{\mathbb{R}^N} |u|^{p+2} dx \leq \frac{p+2}{2|Q|_{L^2}^p} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{Np}{4}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{2(p+2)-Np}{4}}, \quad \forall u \in H^1(\mathbb{R}^3). \quad (1.8)$$

We remark that, similar to [11], one can prove that all optimizers of (1.8) are indeed the scalings and translations of  $Q(x)$ , i.e., belong to the following set

$$\{\lambda Q(\alpha x + y) : \alpha, \beta \in \mathbb{R}^+, y \in \mathbb{R}^N\}. \quad (1.9)$$

Our first theorem gives the existence and uniqueness of minimizers for problem (1.3).

**Theorem 1.1.** (i) When  $0 < p < \frac{4}{N}$ , problem (1.3) has a unique minimizer (up to translations), which is the form of  $u_c = \frac{c\lambda_p^{\frac{N}{2}}}{|Q|_{L^2}^p} Q(\lambda_p x)$ , where  $\lambda_p = \frac{t_p^{\frac{1}{2}}}{c}$  with  $t_p$  being the unique minimum point of the function

$$f_p(t) = \frac{a}{2}t + \frac{b}{4}t^2 - \frac{c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} t^{\frac{Np}{4}}, \quad t \in (0, +\infty). \quad (1.10)$$

(ii) When  $p = \frac{4}{N}$ , problem (1.3) has no minimizer if  $c \leq a^{\frac{N}{4}}|Q|_{L^2}$ . On the contrary, if  $c > a^{\frac{N}{4}}|Q|_{L^2}$ , then (1.3) has a unique minimizer (up to translations)

$$u_c = \frac{c\lambda_p^{\frac{N}{2}}}{|Q|_{L^2}}Q(\lambda_p x) \text{ where } \lambda_p = \frac{(c^{\frac{4}{N}} - a|Q|_{L^2}^{\frac{4}{N}})^{\frac{1}{2}}}{c|Q|_{L^2}^{\frac{N}{2}}b^{\frac{1}{2}}}. \quad (1.11)$$

$$\text{Also, } I(c) = -\frac{(c^{\frac{4}{N}} - a|Q|_{L^2}^{\frac{4}{N}})^2}{4b|Q|_{L^2}^{\frac{N}{2}}}.$$

(iii) When  $\frac{4}{N} < p < \frac{8}{N}$ , problem (1.3) has no minimizer if

$$c < c^* := \left(2|Q|_{L^2}^p \left(\frac{2a}{8-Np}\right)^{\frac{8-Np}{4}} \left(\frac{b}{Np-4}\right)^{\frac{Np-4}{4}}\right)^{\frac{2}{2(p+2)-Np}}. \quad (1.12)$$

On the other hand, if  $c \geq c^*$ , then (1.3) has a unique minimizer (up to translations)

$$u_c = \frac{c\lambda_p^{\frac{N}{2}}}{|Q|_{L^2}}Q(\lambda_p x) \text{ with } \lambda_p = \frac{1}{c} \left(\frac{2(Np-4)a}{(8-Np)b}\right)^{\frac{1}{2}}. \quad (1.13)$$

Moreover, we have  $I(c) = \frac{c^{\frac{2(p+2)-Np}{2}} - c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} \left(\frac{2(Np-4)a}{(8-Np)b}\right)^{\frac{Np}{4}}$  for any  $c \geq c^*$ .

(iv) When  $p \geq \frac{8}{N}$ , problem (1.3) has no minimizer for all  $c > 0$ .

Theorem 1.1 tells that the minimizer of (1.3) must be a scaling of  $Q(x)$ , which extends [12, Theorem 1.1], where the existence of minimizers for (1.3) was discussed. From the above theorem, we see that problem (1.3) has no minimizer if  $p \geq \frac{8}{N}$ . Thus, to obtain the normalized solutions for (1.1), one may search for saddle point for functional (1.4). Stimulated by [6, 1], we investigate the mountain pass type critical point for  $E(\cdot)$  on  $S_c$ .

**Definition 1.1.** Given  $c > 0$ , the functional  $E(\cdot)$  is said to have mountain pass geometry on  $S_c$  if there exists  $K(c) > 0$  such that

$$\gamma(c) := \inf_{h \in \Gamma(c)} \max_{t \in [0,1]} E(h(t)) > \max\{E(h(0)), E(h(1))\} \quad (1.14)$$

holds in the set  $\Gamma(c) = \left\{h \in C([0,1]; S_c) | h(0) \in A_{K(c)} \text{ and } E(h(1)) < 0\right\}$ , where  $A_{K(c)} = \{u \in S_c : |\nabla u|_{L^2}^2 \leq K(c)\}$ .

By studying some analytic properties of  $\gamma(c)$  and involving rigorous arguments, Ye in [12, 13] proved respectively that,

$$\text{either } p > \frac{8}{N}, \text{ or } p = \frac{8}{N} \text{ and } c > c_* := \left(\frac{b|Q|_{L^2}^{\frac{8}{N}}}{2}\right)^{\frac{N}{8-2N}}, \quad (1.15)$$

$E(\cdot)$  possesses the mountain pass geometry on  $S_c$ . Moreover, there exists  $u_c \in S_c$  such that  $E(u_c) = \gamma(c)$ , and  $u_c$  is a solution of (1.1) with some  $\mu \in \mathbb{R}^-$ . Immediately, for the second case, Ye in [14] further studied the asymptotic behavior of  $u_c$  as  $c \rightarrow (c_*)^+$ . Motivated by these results and the proof of our first theorem, we attempt to investigate some properties of problem (1.14) by introducing some new observations and energy estimates. Moreover, as a byproduct, we show that if  $u_c \in S_c$  is critical point of  $E(\cdot)$  on the level  $\gamma(c)$ , then it is unique and indeed a scaling of  $Q(x)$ . Still let  $f_p(\cdot)$  be given by (1.10) and note that it has a unique maximum point in  $(0, +\infty)$  once (1.15) is assumed. Then, we have the following theorem.

**Theorem 1.2.** Assume (1.15) holds and let  $\bar{t}_p$  be the unique maximum point of  $f_p(t)$  in  $(0, +\infty)$ . Then  $\gamma(c) = f_p(\bar{t}_p)$  and it can be attained by  $\bar{u}_c = \frac{c\bar{\lambda}_p^{\frac{N}{2}}}{|Q|_{L^2}}Q(\bar{\lambda}_p x)$ , where  $\bar{\lambda}_p = (\bar{t}_p)^{\frac{1}{2}}/c$ . Also,  $\bar{u}_c$  is a

solution of (1.1) for some  $\mu \in \mathbb{R}^-$ . Moreover,  $\bar{u}_c$  is the unique solution of (1.14) in the following sense: if  $\bar{u} \in S_c$  is a critical point of  $E(\cdot)$  on  $S_c$  and its energy equals to  $\gamma(c)$ , namely,

$$E'(\bar{u})|_{S_c} = 0 \quad \text{and} \quad E(\bar{u}) = \gamma(c). \quad (1.16)$$

Then, up to translations,  $\bar{u} = \bar{u}_c$ .

**Remark 1.2.** If  $p = \frac{8}{N}$  and  $c > c_*$ , one can easily check that  $\bar{t}_p = \frac{a}{b}[(\frac{c}{c_*})^{\frac{8-2N}{N}} - 1]^{-1}$ , we thus deduce from Theorem 1.2 and (1.15) that  $\bar{u}_c = \left(\frac{a^2(cc_*)^{\frac{8-4N}{N}}}{2bc_*^2((\frac{c}{c_*})^{\frac{8-2N}{N}} - 1)^2}\right)^{\frac{N}{8}} Q\left(\left[\frac{a}{bc^2((\frac{c}{c_*})^{\frac{8-2N}{N}} - 1)}\right]^{\frac{1}{2}}x\right)$  and  $\gamma(c) = \frac{1}{4b}[(\frac{c}{c_*})^{\frac{8-2N}{N}} - 1]^{-1}$ . This extends the results of [14] where the asymptotic behaviors of  $\bar{u}_c$  and the value of  $\gamma(c)$  as  $c \rightarrow (c_*)^+$  were studied.

## 2 Proof of Main Results.

In this section, we give the proof of Theorems 1.1 and 1.2 by employing the Gagliardo-Nirenberg inequality (1.8) and some energy estimates. We first remark that by a simple rescaling, one can easily show that

$$I(c) \leq 0 \quad \text{for all } c > 0 \text{ and } 0 < p < 2^* - 2. \quad (2.1)$$

Moreover, utilizing (1.8), we see that for any  $u \in S_c$ ,

$$\begin{aligned} E(u) &\geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2 - \frac{c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{Np}{4}} \\ &= f_p(t) \quad \text{by setting } t = \int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned} \quad (2.2)$$

where  $f_p(\cdot)$  is given by (1.10).

**Proof of Theorem 1.1.** (i). Since  $p < \frac{4}{N}$ , one can easily check that  $f_p(t)$  ( $t \in (0, \infty)$ ) attains its minimum at a unique point, denoted by  $t_p$ . Therefore, we obtain from (2.2) that

$$I(c) = \inf_{u \in S_c} E(u) \geq f_p(t_p). \quad (2.3)$$

On the other hand, set

$$u_\lambda(x) = \frac{c\lambda^{\frac{N}{2}}}{|Q|_{L^2}} Q(\lambda x), \quad (2.4)$$

where  $\lambda > 0$  will be determined later. Then,  $u_\lambda \in S_c$  and it follows from (1.7) that

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 dx = c^2 \lambda^2; \quad \int_{\mathbb{R}^N} |u_\lambda|^{p+2} dx = \frac{(p+2)c^{p+2}\lambda^{\frac{Np}{2}}}{2|Q|_{L^2}^p}.$$

Consequently,

$$E(u_\lambda) = \frac{a}{2} c^2 \lambda^2 + \frac{b}{4} (c^2 \lambda^2)^2 - \frac{c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} (c^2 \lambda^2)^{\frac{Np}{4}} = f_p(c^2 \lambda^2). \quad (2.5)$$

Choosing  $\lambda = t_p^{\frac{1}{2}}/c$ , i.e.,  $c^2 \lambda^2 = t_p$ , it follows from (2.5) that  $I(c) \leq E(u_\lambda) = f_p(t_p)$ . Together with (2.3), we deduce that

$$I(c) = f_p(t_p) = \inf_{t \in \mathbb{R}^+} f_p(t), \quad (2.6)$$

and  $u_\lambda$  with  $\lambda = t_p^{\frac{1}{2}}/c$ , i.e.,  $u_\lambda = u_c = \frac{c}{|Q|_{L^2}} \left(t_p^{\frac{1}{2}}/c\right)^{\frac{N}{2}} Q\left(t_p^{\frac{1}{2}}x/c\right)$  is a minimizer of (1.3).

It remains to prove that, up to translations,  $u_c$  is the unique minimizer of (1.3). Indeed, if  $u_0 \in S_c$  is a minimizer, it then follows from (2.2) that  $E(u_0) \geq f_p(t_0)$ , with  $t_0 := \int_{\mathbb{R}^N} |\nabla u_0|^2 dx$ , where the

“=” holds if and only if  $u_0$  is an optimizer of (1.8). This and (2.6) further imply that  $t_0 = t_p$  and  $f_p(t_0) = E(u_0)$ . Thus,  $u_0$  is an optimizer of (1.8) and it follows from (1.9) that up to translations,  $u_0$  must be the form of  $u_0(x) = \alpha Q(\beta x)$ . Utilizing  $\int_{\mathbb{R}^N} |u_0|^2 dx = c^2$ ,  $\int_{\mathbb{R}^N} |\nabla u_0|^2 dx = t_p$  and (1.7), we see that  $\alpha = \frac{c}{|Q|_{L^2}^{\frac{1}{N}}} \left(\frac{t_p}{c}\right)^{\frac{N}{2}}$  and  $\beta = \frac{t_p}{c}$ , hence,  $u_0 = u_c$ .

(ii). Since  $p = \frac{4}{N}$ , then,

$$f_p(t) = \frac{1}{2} \left( a - \frac{c^{\frac{4}{N}}}{|Q|_{L^2}^{\frac{4}{N}}} \right) t + \frac{b}{4} t^2. \quad (2.7)$$

If  $c \leq a^{\frac{N}{4}} |Q|_{L^2}$  one can easily deduce from (2.2) that  $E(u) \geq f(\int_{\mathbb{R}^N} |\nabla u|^2 dx) > 0$  for all  $u \in S_c$ . In view of (2.1), this indicates that (1.3) has no minimizer. Next, we turn to the case of  $c > a^{\frac{N}{4}} |Q|_{L^2}$ . From (2.7), we know that  $f_p(t) (t \in (0, +\infty))$  attains its minimum at the unique point  $t_p = \frac{c^{\frac{4}{N}} - a|Q|_{L^2}^{\frac{4}{N}}}{b|Q|_{L^2}^{\frac{4}{N}}}$ . Similar to the arguments of part (i), one can prove that, up to translations  $u_c$  given

by (1.11) is the unique minimizer of (1.3) and the energy  $I(c) = f_p(t_p) = -\frac{(c^{\frac{4}{N}} - a|Q|_{L^2}^{\frac{4}{N}})^2}{4b|Q|_{L^2}^{\frac{8}{N}}}$ .

(iii). For the case  $\frac{4}{N} < p < \frac{8}{N}$ , let  $\alpha = \frac{8-Np}{4}$  and  $\beta = 1 - \alpha = \frac{Np-4}{4}$ , it follows from the Young's inequality that, for any  $t > 0$ ,

$$\frac{a}{2} t + \frac{b}{4} t^2 = \alpha \left( \frac{a}{2\alpha} t \right) + \beta \left( \frac{b}{4\beta} t^2 \right) \geq \left( \frac{a}{2\alpha} \right)^\alpha \left( \frac{b}{4\beta} \right)^\beta t^{\alpha+2\beta} = \left( \frac{2a}{8-Np} \right)^{\frac{8-Np}{4}} \left( \frac{b}{Np-4} \right)^{\frac{Np-4}{4}} t^{\frac{Np}{4}}$$

where the “=” in the second inequality holds if and only if  $\frac{a}{2\alpha} t = \frac{b}{4\beta} t^2$ , i. e.,  $t = t_0 := \frac{2\beta a}{\alpha b} = \frac{2(Np-4)a}{(8-Np)b}$ . In view of (2.2) and noting that  $c_*$  is given by (1.12), we therefore have

$$E(u) \geq \frac{c^{*\frac{2(p+2)-Np}{2}} - c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} t_0^{\frac{Np}{4}} = f_p(t_0) \text{ for all } u \in S_c. \quad (2.8)$$

If  $c < c_*$ , we then deduce from (2.8) that  $E(u) > 0$  for all  $u \in S_c$ . Thus, problem (1.3) cannot be achieved for (2.1).

If  $c \geq c_*$ , on the one hand, we deduce from (2.8) that  $I(c) \geq f_p(t_0)$ . On the other hand, let  $u_\lambda(x)$  be as in (2.4) and set  $\lambda = t_0^{\frac{1}{2}}/c$ , then  $I(c) \leq E(u_\lambda) = f_p(t_0)$ . This indicates that  $u_\lambda$  is a minimizer of (1.3) and  $I(c) = f_p(t_0) = \frac{c^{*\frac{2(p+2)-Np}{2}} - c^{\frac{2(p+2)-Np}{2}}}{2|Q|_{L^2}^p} \left( \frac{2(Np-4)a}{(8-Np)b} \right)^{\frac{Np}{4}}$  for any  $c \geq c_*$ . The uniqueness of minimizers can be proved by the same argument of part (i).

(iv). If  $p > \frac{8}{N}$ , or  $p = \frac{8}{N}$  and  $c > \left( \frac{b|Q|_{L^2}^{\frac{8}{N}}}{2} \right)^{\frac{N}{8-2N}}$ , it follows (2.4) and (2.5) that  $I(c) \leq \lim_{\lambda \rightarrow +\infty} E(u_\lambda) = -\infty$ , and thus problem (1.3) cannot be attained. On the other hand, if  $p = \frac{8}{N}$  and  $c \leq \left( \frac{b|Q|_{L^2}^{\frac{8}{N}}}{2} \right)^{\frac{N}{8-2N}}$ , from (2.2) we have  $E(u) > 0$  for all  $u \in S_c$ . This together with (2.1) obviously indicates that problem (1.3) cannot be attained.  $\square$

**Proof of Theorem 1.2.** Firstly, similar to the proof of [13, Lemma 3.1], from the definition 1.1, one can prove that there exists  $K(c) > 0$  which can be chosen small enough such that,  $E(\cdot)$  admits mountain pass geometry on  $S_c$  if (1.15) is assumed. In what follows, we thus always assume that  $K(c) < \bar{t}_p$ , where  $\bar{t}_p$  denotes the unique maximum point of  $f_p(t)$  in  $(0, +\infty)$ .

For any  $h(s) \in \Gamma(c)$ , one can deduce from (2.2) that

$$E(h(s)) \geq f_p \left( \int_{\mathbb{R}^N} |\nabla h(s)|^2 dx \right), \quad (2.9)$$

where “=” holds if and only if  $h(s) \in S_c$  is an optimizer of (1.8), i. e., up to translations,

$$(h(s))(x) = \frac{c\alpha^{\frac{N}{2}}}{|Q|_{L^2}} Q(\alpha x) \text{ for some } \alpha > 0. \quad (2.10)$$

Since  $h(0) \in A_{K(c)}$  with  $K(c) < \bar{t}_p$ , and note that  $f_p(t) > 0 \forall t \in (0, \bar{t}_p]$ , we thus have

$$\int_{\mathbb{R}^N} |\nabla h(0)|^2 dx < \bar{t}_p < \int_{\mathbb{R}^N} |\nabla h(1)|^2 dx. \quad (2.11)$$

As a consequence of (2.9) and (2.11), there holds that

$$\max_{s \in [0,1]} E(h(s)) \geq f_p(\bar{t}_p) = \max_{t \in \mathbb{R}^+} f_p(t). \quad (2.12)$$

Thus,

$$\gamma(c) \geq f_p(\bar{t}_p). \quad (2.13)$$

On the contrary, let  $u_\lambda(x)$  be the trial function given by (2.4) with  $\lambda = \bar{\lambda}_p = (\bar{t}_p)^{\frac{1}{2}}/c$ . Set  $\bar{h}(s) := s^{\frac{N}{4}} u_\lambda(s^{\frac{1}{2}} x)$ , then one can check that  $E(\bar{h}(s)) = f_p(\bar{t}_p s)$ . Choosing  $0 < \tilde{t}_p < \bar{t}_p$  small enough such that  $\bar{h}(\tilde{t}_p/\bar{t}_p) \in A_{K(c)}$ , and  $\hat{t}_p > \bar{t}_p$  such that  $f_p(\hat{t}_p) < 0$ , let  $h(s) = \bar{h}((1-s)\tilde{t}_p/\bar{t}_p + \hat{t}_p s/\bar{t}_p)$ . Then,  $h(0) = \bar{h}(\tilde{t}_p/\bar{t}_p) \in A_{K(c)}$  and  $E(h(1)) = E(\bar{h}(\hat{t}_p/\bar{t}_p)) = f_p(\hat{t}_p) < 0$ . This indicates that  $h \in \Gamma(c)$ , and

$$\gamma(c) \leq \max_{t \in [0,1]} E(h(t)) = E(u_{\bar{\lambda}_p}) = f_p(\bar{t}_p).$$

Combing with (2.13), we deduce that  $\gamma(c) = f_p(\bar{t}_p)$  and  $u_{\bar{\lambda}_p} = \bar{u}_c(x) = \frac{c}{|Q|_{L^2}} \left( \bar{t}_p^{\frac{1}{2}}/c \right)^{\frac{N}{2}} Q(\bar{t}_p^{\frac{1}{2}} x/c)$  is a solution of problem (1.14).

We next prove that  $\bar{u}_c$  satisfies equation (1.1) for some  $\mu \in \mathbb{R}^-$ . Actually, in view of  $f'_p(\bar{t}_p) = 0$  and  $\bar{\lambda}_p = (\bar{t}_p)^{\frac{1}{2}}/c$ , we have

$$\frac{Npc^{\frac{2(p+2)-Np}{2}}}{4|Q|_{L^2}^p} (\bar{t}_p)^{\frac{Np-4}{4}} = a + b\bar{t}_p = a + b \int_{\mathbb{R}^N} |\nabla \bar{u}_c|^2 dx. \quad (2.14)$$

Moreover, since  $Q(x)$  is a solution of (1.6) and note that  $\bar{\lambda}_p = (\bar{t}_p)^{\frac{1}{2}}/c$ , it follows that  $\bar{u}_c$  satisfies

$$-\frac{Npc^{\frac{2(p+2)-Np}{2}}}{4|Q|_{L^2}^p} (\bar{t}_p)^{\frac{Np-4}{4}} \Delta \bar{u}_c - |\bar{u}_c|^p \bar{u}_c = -\frac{(4+2p-Np)(c\bar{\lambda}_p^{\frac{N}{2}})^p}{4|Q|_{L^2}^p} \bar{u}_c.$$

This together with (2.14) indicates that  $\bar{u}_c$  is a solution of (1.1) with  $\mu = -\frac{(4+2p-Np)(c\bar{\lambda}_p^{\frac{N}{2}})^p}{4|Q|_{L^2}^p}$ .

We finally prove that, up to translations,  $\bar{u}_c$  is the unique solution of  $\gamma(c)$ . Suppose  $\bar{u}$  is a solution of  $\gamma(c)$  and satisfies (1.16), then there exists  $\mu \in \mathbb{R}$  such that  $E'(u) = \mu u$ , it then follows from the Nehari and Pohozaev identity (see e.g., [7, Lemma 2.1]) that

$$\frac{a}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx + \frac{b}{2} \left( \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx \right)^2 - \frac{pN}{4(p+2)} \int_{\mathbb{R}^N} |\bar{u}|^{p+2} dx = 0. \quad (2.15)$$

Let  $\hat{h}(s) := s^{\frac{N}{4}} \bar{u}(s^{\frac{1}{2}} x)$  and

$$g(s) := E(\hat{h}(s)) = \frac{as}{2} \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx + \frac{bs^2}{4} \left( \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx \right)^2 - \frac{s^{\frac{pN}{4}}}{p+2} \int_{\mathbb{R}^N} |\bar{u}|^{p+2} dx. \quad (2.16)$$

(2.15) indicates that  $g(s)$  ( $s \in (0, +\infty)$ ) attains its maximum at the unique point  $s = 1$ , and  $\lim_{s \rightarrow +\infty} g(s) = -\infty$ . Choosing  $0 < \tilde{s} < 1 < \hat{s}$  such that  $\hat{h}(\tilde{s}) \in A_{K(c)}$  and  $g(\hat{s}) < 0$ , then  $h_0(s) := \hat{h}((1-s)\tilde{s} + \hat{s}s) \in \Gamma(c)$  and  $\max_{s \in [0,1]} E(h_0(s)) = E(\bar{u})$ . As the arguments of (2.9) and (2.12), we see that

$$f_p(\bar{t}_p) = \gamma(c) = E(\bar{u}) = \max_{s \in [0,1]} E(h_0(s)) \geq \max_{t \in \mathbb{R}^+} f_p(t) = f_p(\bar{t}_p).$$

Together with (2.10), this means that  $\bar{u}$  must be the form of  $\frac{c\alpha^{\frac{N}{2}}}{|Q|_{L^2}} Q(\alpha x)$  for some  $\alpha > 0$ . Take it into the equality  $f_p(\bar{t}_p) = E(\bar{u})$ , we further obtain that  $\alpha = \bar{\lambda}_p$  and  $\bar{u} = \bar{u}_c$ .  $\square$

**Acknowledgements:** X. Y. Zeng is supported by NSFC grant 11501555, and Y. M. Zhang is supported by NSFC grant 11471330. This work is also partially supported by the Fundamental Research Funds for the Central Universities(WUT: 2017 IVA 075 and 2017 IVA 076).

## References

- [1] J. Bellazzini, L. Jeanjean and T. J. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations, *Proc. Lond. Math. Soc.* **107**(3) (2013) 303–339.
- [2] Y. B. Deng, S. J. Peng and W. Shuai, Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in  $\mathbb{R}^3$ . *J. Funct. Anal.* **269** (2015) 3500–3527.
- [3] H. L. Guo and Y. B. Wang, A remark about a constrained variational problem, preprint.
- [4] Y. He, G. B. Li and S. J. Peng, Concentrating bound states for Kirchhoff type problems in  $\mathbb{R}^3$  involving critical Sobolev exponents, *Adv. Nonlinear Stud.* **14** (2014) 441–468.
- [5] X. M. He and W. M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in  $\mathbb{R}^3$ , *J. Differential Equations* **2** (2012) 1813–1834.
- [6] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal. Theory T. M. A.* **28** (1997) 1633–1659.
- [7] L. Jeanjean and T. J. Luo, Sharp nonexistence results of prescribed  $L^2$ -norm solutions for some class of Schrödinger-Poisson and quasi-linear equations, *Z. Angew. Math. Phys.* **64** (4) (2013) 937–954.
- [8] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
- [9] J. L. Lions, On some equations in boundary value problems of mathematical physics, in: *Contemporary Developments in Continuum Mechanics and Partial Differential Equations* (Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), in: *North-Holland Math. Stud.*, vol. 30, North-Holland, Amsterdam, 1978, 284–346.
- [10] S. I. Pohozaev, A certain class of quasilinear hyperbolic equations, *Mat. Sb.* **96** (1975) 152–168.
- [11] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, *Comm. Math. Phys.*, **87** (1983), 567–576.
- [12] H. Y. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, *Math. Methods Appl. Sci.* **38** (2015), 2663–2679.
- [13] H. Y. Ye, The existence of normalized solutions for  $L^2$ -critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.* **66** (2015), 1483–1497.
- [14] H. Y. Ye, The mass concentration phenomenon for  $L^2$ -critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.* **67** (2): 29 (2016).